

DOWKER-TYPE THEOREMS FOR HYPERCONVEX DISCS

G. FEJES TÓTH AND F. FODOR

ABSTRACT. A hyperconvex disc of radius r is a planar set with nonempty interior that is the intersection of closed circular discs of radius r . A convex disc-polygon of radius r is a set with nonempty interior that is the intersection of a finite number of closed circular discs of radius r . We prove that the maximum area and perimeter of convex disc- n -gons of radius r contained in a hyperconvex disc of radius r are concave functions of n , and the minimum area and perimeter of disc- n -gons of radius r containing a hyperconvex disc of radius r are convex functions of n . We also consider hyperbolic and spherical versions of these statements.

1. INTRODUCTION AND RESULTS

Let K denote a *convex disc*, that is, a compact convex set with non-empty interior in the Euclidean plane \mathbb{E}^2 . Confirming a conjecture of Kershner, Dowker [9] proved that the maximum area of n -gons inscribed in K is a concave function of n , while the minimum area of n -gons circumscribed about K is a convex function of n . Dowker observed that the argument of his proof also shows that if K is centrally symmetric, then among the $2n$ -gons of maximum area inscribed in K , as well as among the $2n$ -gons of minimum area circumscribed about K , there is one that is centrally symmetric with the same centre as K .

Dowker's theorems play an important role in the theory of packing and covering in the Euclidean plane. For example, they are essential to L. Fejes Tóth's proof about the packing density of convex discs in a convex hexagon, cf. [12]. L. Fejes Tóth [13], Molnár [20], and Eggleston [10] observed independently of each other that Dowker's results remain true if the word "area" is replaced by "perimeter". The results about centrally symmetric discs was generalized in [11] for kn -gons inscribed in and circumscribed about a convex disc with k -fold rotational symmetry.

Convex discs in the plane are intersections of half-planes. As a natural strengthening of convexity we study sets that are the intersection of discs of radius r . Properties of such sets were studied in several papers and different authors used different names for them. It appears that it was Mayer [19] who first investigated such sets. He studied them in the more general setting of Minkowski geometry and called them "überkonvex". He characterized such sets with the property that together with any two points of them the shorter arcs of both circles of radius r connecting the two points belong to the set. Mayer's paper inspired further research in the 1930s and the 1940s (cf. for example Blanc [5], Buter [6], Pasqualini [21], Santaló [22], van der Corput [23], Vincensini [24], see also the survey paper by Danzer, Grünbaum, and Klee [8]). More than seven decades after Mayer, Bezdek

Appeared in *Period. Math. Hungar.* **70** (2015), 131–144. DOI 10.1007/s10998-014-0071-y.

The final publication is available at <https://link.springer.com/article/10.1007%2Fs10998-014-0071-y>.

et al. [4] and Kupitz et al. [17, 18] made a detailed investigation of such sets in \mathbb{E}^n (for more information see also [1] and [2]). Bezdek et al. [4] and Kupitz et al. [17, 18] call such sets spindle convex, while L. Fejes Tóth [15, 16], who proved packing and covering theorems for them, used the term r -convex. We use the English translation of Mayer's phrase. A planar set is *hyperconvex with radius r* if it is the intersection of circular discs of radius r . Note that if we consider half-planes as circles of infinite radius, then we obtain the family of linearly convex sets for $r = \infty$. A *hyperconvex disc of radius r* is a compact hyperconvex set with radius r and with nonempty interior. A *disc-polygon of radius $r < \infty$* is the intersection of a finite number of discs of radius r in such a way that the interior of the set is nonempty. We shall assume that whenever a disc-polygon is represented as the intersection of some discs, each disc is essential, that is, if discarded, then the intersection changes. Thus, the boundary of a disc-polygon of radius r consists of a finite number of radius r circular arcs of positive length, called *sides*, each of which is part of the boundary of a unique generating disc. The sides follow in a natural cyclic order on the boundary of the disc-polygon. Two consecutive sides in this order meet in a *vertex*, except in the case of disc-2-gons in which two consecutive sides intersect in a pair of vertices. Therefore, if a disc-polygon of radius r is the intersection of $n > 1$ essential discs, then it has exactly n sides and n vertices. We will call such sets *disc- n -gons of radius r* .

In this article we extend Dowker's theorem for the case when K is a hyperconvex disc of radius r and the approximating objects are disc-polygons of radius r . Since the value of r is fixed throughout the article, we suppress its notation unless this omission may cause confusion.

Let $a_i(n)$ and $p_i(n)$ denote the maximum area and maximum perimeter of convex disc-polygons with at most n vertices contained in K . Let $a_c(n)$ and $p_c(n)$ denote the minimum area and minimum perimeter of convex disc-polygons with at most n vertices containing K . It is easily seen that $a_i(n)$ and $p_i(n)$ are realized by disc-polygons that are inscribed in K in the sense that their vertices are on the boundary of K . Similarly, $a_c(n)$ and $p_c(n)$ are realized by disc-polygons that are circumscribed about K meaning that their sides are tangential to the boundary of K .

Theorem 1. *We have, for $n \geq 4$,*

- i) $a_c(n-1) + a_c(n+1) \geq 2a_c(n)$,
- ii) $p_c(n-1) + p_c(n+1) \geq 2p_c(n)$,
- iii) $a_i(n-1) + a_i(n+1) \leq 2a_i(n)$,
- iv) $p_i(n-1) + p_i(n+1) \leq 2p_i(n)$.

Theorem 2. *If K has k -fold rotational symmetry, then there are disc-polygons P_{ca} and P_{cp} with at most kn vertices circumscribed about K , as well as disc-polygons P_{ia} and P_{ip} with at most kn vertices inscribed in K such that all of them have k -fold rotational symmetry with the same centre as K and $\text{area}(P_{ca}) = a_c(kn)$, $\text{per}(P_{cp}) = p_c(kn)$, $\text{area}(P_{ia}) = a_i(kn)$, and $\text{per}(P_{ip}) = p_i(kn)$.*

We note that the above theorems were proved by Bezdek et al. in [4] for the special case when K is a closed circular disc of radius $r < 1$. We stated Theorem 1 for $n \geq 4$, so that it includes the case $r = \infty$, as well. If $r < \infty$, then also digons can occur and Theorem 1 holds for $n \geq 3$.

2. PROOFS

Since the case $r = \infty$ of our theorems is well-known, we shall restrict our attention to the case when r is finite. We will use *circle-polygons of radius r* . Following Bezdek et al. (cf. [4], page 224) we define such circle-polygons as described below. Let v_1, \dots, v_n be a sequence of points such that $d(v_i, v_{i+1}) \leq 2r$ for $i = 1, \dots, n$ and with the convention that $v_{n+1} = v_1$. Let $v_i v_{i+1}$ denote one of the shorter circular arcs of radius r incident with v_i and v_{i+1} . The union of the arcs $v_1 v_2, \dots, v_n v_1$ is called a *circle-polygon of radius r* , which we denote by C . We call the points v_i the *vertices* and the circular arcs $v_i v_{i+1}$ the *sides* of C . Note that a circle-polygon of radius r does not necessarily bound a convex domain and it may be self-intersecting. The boundary of a convex disc-polygon of radius r is an example of circle-polygons of radius r . Since r is fixed, henceforth we omit it from our notation.

We fix an orientation of the plane which induces a natural orientation of any circular arc. Thus, a circular arc has a well-defined initial and terminal point. We say that a circle-polygon is orientable (or proper) if the induced orientation of its sides determines an orientation of the entire circle-polygon. Disc-polygons are obviously orientable. In this paper we only consider orientable circle-polygons. Subsequently, we will omit the word “orientable”.

Let $P = \{C_1, \dots, C_n\}$ be a finite collection of closed circle-polygons. We call such a collection a *multiple circle-polygon*. It encloses a *signed multiset* in which the multiplicity of a point p is defined as $\sum_{i=1}^n \chi_i(p)$, where $\chi_i(p)$ is the winding number of C_i around p while going around C_i in the positive direction. The area and the perimeter of P are defined as

$$\text{area}(P) = \sum_{k \in \mathbb{Z}} k \cdot \text{area}(\{p : \chi(p) = k\}),$$

and

$$\text{per}(P) = \sum_{i=1}^n \text{per}(C_i),$$

respectively. The intersection of the discs corresponding to the sides of the circle-polygons C_1, \dots, C_n is the *core* of P . In the proofs of Theorem 1 and 2 *double circle-polygons* play a crucial role. These are those multiple circle-polygons for which the core is not empty and the points of the core have multiplicity 2.

Following Bezdek, Csikós, and Connelly (cf. [3], p. 55) and Bezdek et al. (cf. [4], p. 203), we introduce the *arc-distance of radius r* of two points. More precisely, let $x, y \in \mathbb{E}^2$ such that $d(x, y) \leq 2r$. Then their arc-distance of radius r , denoted by $\varrho_r(x, y)$, is defined as the length of the shorter circular arc of radius r connecting x and y . It is noted in [3] that the arc-distance is not a metric. In fact, its behaviour is described in the following two statements from [3], which we cite in slightly modified forms.

Proposition 1 (Bezdek, Csikós, and Connelly [3], Lemma 1). *Let $r > 0$ and $x, y, z \in \mathbb{E}^2$ such that $d(x, y) \leq 2r$, $d(x, z) \leq 2r$, and $d(y, z) \leq 2r$. Let I denote the intersection of the two discs of radius r whose boundary contains x and z . Then $\varrho_r(x, y) + \varrho_r(y, z) > \varrho_r(x, z)$, $\varrho_r(x, y) + \varrho_r(y, z) = \varrho_r(x, z)$ or $\varrho_r(x, y) + \varrho_r(y, z) < \varrho_r(x, z)$ according as $y \notin I$, $y \in \text{bd } I$ or $y \in \text{int } I$.*

Proposition 2 (Bezdek, Csikós, and Connelly [3], Lemma 2). *Let $r > 0$ and $x, y, z, w \in \mathbb{E}^2$ be the vertices of a disc-quadrilateral of radius r , listed in a cyclic*

order. Then

$$\varrho_r(x, y) + \varrho_r(z, w) < \varrho_r(x, z) + \varrho_r(y, w),$$

that is, the sum of the arc-lengths of the diagonals is larger than the sum of the arc-lengths of any two nonadjacent sides.

We begin with the proofs of i) and ii) of Theorem 1 and the corresponding cases of Theorem 2.

Let K be a hyperconvex disc and P a double circle-polygon circumscribed about K . Let s_1, \dots, s_n be the sides of P . We choose the notation so that s_{i-1} and s_i are neighbours meeting in a vertex v_i . If P is a single circle-polygon winding twice around the core of P , this can be achieved by the convention $s_n = s_0$ and $s_{n+1} = s_1$. On the other hand, if P is the union of two circle-polygons, with l and m sides, say, then we label the sides of the l -gon by $s_1 = s_{l+1}, s_2, \dots, s_l = s_0$ and the sides of the m -gon by $s_{l+3} = s_{l+m+3}, s_{l+4}, \dots, s_{l+m+2} = s_{l+2}$. Let C_i denote the circle containing the side s_i , o_i its centre and D_i the discs bounded by C_i . To each side of P we assign a point $t_i \in s_i \cap \text{bd } K$. If $s_i \cap \text{bd } K$ is not a single point, then we can choose t_i arbitrarily with the restriction that if $C_i = C_j$ then $t_i = t_j$.

We note that to any permutation of the points t_i corresponds a double circle-polygon in which the side associated with t_i is the arc of C_i lying in the intersection of the discs associated with the points preceding and following t_i in the permutation. Arrange the points t_i in the order as they follow on $\text{bd } K$ while going around in the positive direction. As there might be coincident points among the t_i 's, we have to specify the order of such points. If $t_i = t_j$ and $C_i = C_j$, then the order of t_i and t_j can be chosen arbitrarily. Suppose that $t_i = t_j$ but $C_i \neq C_j$. If the arc $v_{i-1}t_i$ is contained in D_j then t_i precedes t_j , and if the arc $v_{j-1}t_j$ is contained in D_i then t_j precedes t_i .

Enumerate the points t_i by taking every second from the above natural order and let P^* be the double circle-polygon corresponding to this order. The statements of our theorems for circumscribed disc-polygons are immediate corollaries of the following

Lemma 1. *We have*

$$\text{area}(P) \geq \text{area}(P^*) \quad \text{and} \quad \text{per}(P) \geq \text{per}(P^*).$$

Note that if K is a disc-polygon with at most n sides, then inequalities i) and ii) of Theorem 1 are obvious. Thus, we may assume that K is not a disc-polygon with at most n sides. Then there are disc- l -gons circumscribed about K for all $l \leq n+1$. Now, if P is the union of two simple circle-polygons Q and R circumscribed about K with $n-1$ and $n+1$ sides, respectively, then P^* consists of two simple circle-polygons Q' and R' with n sides. Thus

$$\begin{aligned} \text{area}(Q) + \text{area}(R) &= \text{area}(P) \geq \text{area}(P^*) = \text{area}(Q') + \text{area}(R') \\ &\geq 2 \min\{\text{area}(Q'), \text{area}(R')\}, \end{aligned}$$

and

$$\begin{aligned} \text{per}(Q) + \text{per}(R) &= \text{per}(P) \geq \text{per}(P^*) = \text{per}(Q') + \text{per}(R') \\ &\geq 2 \min\{\text{per}(Q'), \text{per}(R')\}. \end{aligned}$$

Suppose now that K has k -fold rotational symmetry. Again, we may assume that K is not a disc-polygon with at most kn vertices, as otherwise the statement of

Theorem 2 is obvious. Consider a disc-polygon Q_0 with nk vertices circumscribed about K . The rotations through $2\pi/j$, $j = 1, \dots, k-1$ carrying K onto itself transform Q_0 into Q_1, \dots, Q_{k-1} . As above, we associate with each side of these disc-polygons a point of tangency with $\text{bd } K$. Any pair Q_i, Q_j , $0 \leq i < j \leq k-1$, forms a double circle-polygon P . If P does not coincide with the corresponding P^* , then we replace Q_i and Q_j by the two disc-polygons forming P^* . Obviously, the new polygons have nk vertices. By this replacement neither the total area nor the total perimeter of the disc-polygons Q_0, \dots, Q_{k-1} increases. We repeat this process until all double circle-polygons P formed by a pair Q_i, Q_j coincide with the corresponding P^* .

Let T be the union of the selected points of tangency of all sides of Q_0, \dots, Q_{k-1} . Let us order the elements of T cyclically on $\text{bd } K$. T has k^2n elements, and by construction, it has k -fold rotational symmetry. Therefore a rotation around the centre of K by an angle of $2\pi/k$ carries each element of T in the kn -th element following it in the cyclic order. The disc-polygons resulting from the above procedure have the property that between any two tangency points of consecutive sides there are exactly $k-1$ tangency points of sides of other Q_i 's. Thus any of these disc-polygons arises by taking the intersection of the discs corresponding to every k -th tangency point. Therefore all of them have k -fold rotational symmetry.

Proof of Lemma 1. With each vertex v_i of P we associate the arc $a_i = \widehat{t_{i-1}t_i}$ of $\text{bd } K$ between t_{i-1} and t_i . If $P = P^*$, then the statement of Lemma 1 is obvious. If $P \neq P^*$, then for some i and j we have $a_j \subset a_i$. Let C_{j-1} intersect s_i in v'_j and let C_j intersect s_{i-1} in v'_i . We obtain a new double circle-polygon P' by replacing the sides s_{i-1} , s_i , s_{j-1} , and s_j by the arcs $v_{i-1}v'_i$, v'_jv_{i+1} , $v_{j-1}v'_j$, and v'_iv_{j+1} , respectively, so that the cyclic order of the vertices of P' is $\dots v_{i-2}v_{i-1}v'_iv_{i+1} \dots v_{j-1}v'_jv_{j+1}v_{i+2} \dots v_{i-2}$. Let t denote the area of the region R enclosed by the arcs $v_iv'_j$, v'_jv_j , $v_jv'_i$, and v'_iv_i , where we allow that R is degenerate and its area is zero (see Figure 1).

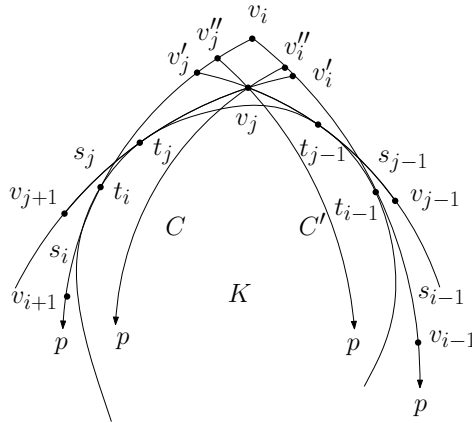


FIGURE 1.

Then

$$\text{area}(P) = \text{area}(P') + t \geq \text{area}(P').$$

We show that also

$$\text{per}(P) \geq \text{per}(P').$$

We have

$$\text{per}(P') = \text{per}(P) + \varrho_r(v'_j, v_j) + \varrho_r(v_j, v'_i) - \varrho_r(v'_i, v_i) - \varrho_r(v_i, v'_j),$$

thus we have to show that

$$\varrho_r(v'_i, v_i) + \varrho_r(v_i, v'_j) \geq \varrho_r(v'_j, v_j) + \varrho_r(v_j, v'_i).$$

Let p be the point of intersection of C_{i-1} and C_i different from v_i . Let C and C' be the circles of radius r passing through the points v_j and p . Denote their centres with o and o' , respectively. Since $p \in D_{j-1} \cap D_j$, one of these circles, say C intersects the arc $v'_i v_i$, while C' intersects the arc $v_i v'_j$. Write $v''_i = C \cap v'_i v_i$ and $v''_j = C' \cap v_i v'_j$.

Observe that $\angle po_i v_i = \pi - \angle o_{i-1} p o_i$, $\angle po_i v''_j = \pi - \angle o_i p o'$, $\angle pov''_i = \pi - \angle o_{i-1} p o$, and $\angle pov_j = \pi - \angle o p o'$. Hence

$$(1) \quad \angle v_i o v''_j = \angle po_i v_i - \angle po_i v''_j = \angle o_i p o' - \angle o_{i-1} p o_i = \angle o_{i-1} p o'$$

and

$$(2) \quad \angle v''_i o v_j = \angle pov''_i - \angle pov_j = \angle o' p o - \angle o_{i-1} p o = \angle o_{i-1} p o'.$$

It follows that $\varrho_r(v_i, v''_j) = \varrho_r(v''_i, v_j)$. In the same way we see that $\varrho_r(v_i, v''_i) = \varrho_r(v''_j, v_j)$.

Obviously, v''_i is not contained in the interior of the intersection of the two circles of radius r passing through v_i and v'_i . It follows by Proposition 1 that $\varrho_r(v_j, v'_i) \leq \varrho_r(v_j, v''_i) + \varrho_r(v'_i, v''_i)$. Similarly we have $\varrho_r(v_j, v'_j) \leq \varrho_r(v_j, v''_j) + \varrho_r(v'_j, v''_j)$. Therefore

$$\begin{aligned} \varrho_r(v'_i, v_i) + \varrho_r(v_i, v'_j) &= \varrho_r(v'_i, v''_i) + \varrho_r(v''_i, v_i) + \varrho_r(v'_j, v''_j) + \varrho_r(v''_j, v_j) \\ &= \varrho_r(v'_i, v''_i) + \varrho_r(v_j, v''_i) + \varrho_r(v'_j, v''_j) + \varrho_r(v_j, v''_j) \\ &\geq \varrho_r(v'_j, v_j) + \varrho_r(v_j, v'_i), \end{aligned}$$

as claimed.

We obtain a sequence of double circle-polygons with non-increasing areas and perimeters by iterating this construction until there are no vertices with the property that the arc of $\text{bd } K$ assigned to one contains the arc assigned to the other. Thus the result of this process is P^* . \square

Now we turn to the proof of the statements of our theorems dealing with inscribed disc-polygons. Let P be a double circle-polygon inscribed in K with sides s_1, \dots, s_n . Again, we assume that s_{i-1} and s_i are neighbours meeting in the vertex v_i and we index the sides of P in the same way as described before Lemma 1. Our goal is to find a new double circle-polygon inscribed in K whose area and perimeter is not less than that of P . We do not consider cases when the area and perimeter of P can be increased in an obvious way. We assume that the vertices of P are not all on an open arc of the boundary of K lying between points of tangency of two parallel supporting lines.

Arrange the vertices of P in the order as they follow on the boundary of K . If two vertices coincide, then we may order them arbitrarily. Observe that for two points that are second neighbours in this cyclic order one of the circles passing through the two points contains the core of P . By taking every second vertex and

connecting them by the shorter arc of the respective circle we obtain a new double circle-polygon P^* inscribed in K .

Lemma 2. *We have*

$$\text{area}(P) \leq \text{area}(P^*) \quad \text{and} \quad \text{per}(P) \leq \text{per}(P^*).$$

From Lemma 2 the statements of our theorems for the inscribed case follow by essentially the same arguments as above. There is one important difference. The union of two simple circle-polygons circumscribed about K is always a double circle-polygon. For inscribed circle-polygons this is not the case. In order to ensure that we get a double circle-polygon we have to start with circle-polygons which have maximum area or maximum perimeter for the given number of sides. Then the vertices of such a circle-polygon cannot all be on an open arc of the boundary of K lying between points of tangency of two parallel supporting lines. This guarantees that the core of the multiple circle-polygon formed by two such circle-polygons is not empty.

Proof of Lemma 2. With each side $s_i = v_{i-1}v_i$ of P , we associate the arc $a_i = \widehat{v_{i-1}v_i}$ of $\text{bd } K$ between v_{i-1} and v_i . If $P = P^*$, then the statement of Lemma 2 is obvious. If $P \neq P^*$, then $a_j \subset a_i$ for some i and j . Let P' denote the double circle-polygon inscribed in K whose vertices, listed in positive cyclic order, are the following:

$$\dots v_{j-2}v_{j-1}v_iv_{i+1}\dots v_{i-1}v_jv_{j+1}\dots$$

Applying Proposition 2 to the circle-quadrangle $v_{i-1}v_{j-1}v_jv_i$ (which may degenerate into a circle-triangle), we obtain that

$$\text{per}(P') \geq \text{per}(P).$$

We need to show that

$$\text{area}(P') \geq \text{area}(P)$$

as well.

We begin the proof of this inequality with a technical statement. Consider three parallel lines L_1 , L_2 and M such that the distance between L_1 and L_2 is less than $2r$ and M is at equal distance from L_1 and L_2 . Let C be a circle of radius r intersecting both L_1 and L_2 . We choose a coordinate system with origin in M so that the points of intersection of L_1 and C are $u = (-x_0, -y_0)$ and $v = (x_0, -y_0)$. For a point $w = (x, y_0)$ inside C we define the region $T = T(x)$ bounded by the circular arcs of radius r connecting the points u and v , v and w and w and u , respectively, so that the arc between u and v and the arc between v and w are outside the triangle $\Delta = uvw$, while the arc between u and w is on the same side of the line uw as Δ (see Figure 2).

We are going to show that

$$(*) \quad \text{area}(T(x)) \text{ is a monotonically decreasing function of } x.$$

Let $S(u, v)$, $S(v, w)$ and $S(w, u)$ denote the circular segments determined by $T(x)$, respectively, as shown on Figure 2. Then

$$\text{area}(T(x)) = \text{area}(\Delta) + \text{area}(S(u, v)) + \text{area}(S(v, w)) - \text{area}(S(u, w)).$$

Clearly, $\text{area}(\Delta)$ and $\text{area}(S(u, v))$ are independent of x , so we have to show the monotonicity of $\text{area}(S(v, w)) - \text{area}(S(u, w))$. This is obvious in the case when

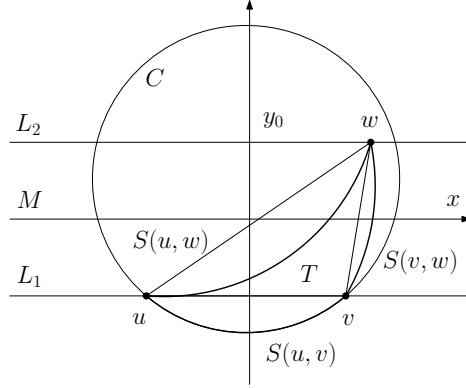


FIGURE 2.

$x \in [-x_0, x_0]$ since in this interval $\text{area}(S(u, w))$ is monotonically increasing and $\text{area}(S(v, w))$ is monotonically decreasing. Observe that

$$\text{area}(T(x)) + \text{area}(T(-x)) = 2 \text{area}(\Delta) + 2 \text{area}(S(u, v)) = \text{constant}.$$

Thus we may assume that $x \geq x_0$.

Let $s(l)$ denote the area of a circular segment cut off from a circle of radius r by a chord of length l . Further define the function $l(z)$ for $z \geq 0$ as the distance from the point $(x, -y_0)$ to $(x + z, y_0)$. It is easily seen that both $s(l)$ and $l(z)$ are monotonically increasing convex functions. It follows that the compound function $f(z) = s(l(z))$ is also monotonically increasing and convex on its domain. We have

$$\text{area}(S(v, w)) - \text{area}(S(u, w)) = f(x + x_0) - f(x - x_0).$$

Now the statement in (*) follows by observing that the right hand side increases by the convexity of f .

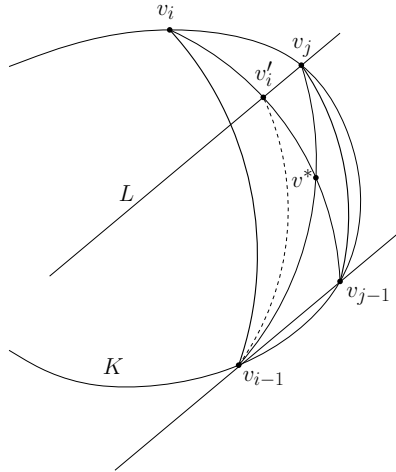


FIGURE 3.

Let v^* be the intersection point of the circular arcs $v_{j-1}v_i$ and $v_{i-1}v_j$ as shown on Figure 3. In order to prove the inequality $\text{area}(P') \geq \text{area}(P)$ we have to

show that the area of the region bounded by the arcs $v_i v_{i-1}$, $v_{i-1} v^*$ and $v^* v_i$ is not smaller than the area of the region bounded by the arcs $v_j v_{j-1}$, $v_{j-1} v^*$ and $v^* v_j$. Equivalently, we can compare the area of the region R_1 bounded by the arcs $v_{i-1} v_{j-1}$, $v_{j-1} v_i$ and $v_i v_{i-1}$ to that of R_2 bounded by the arcs $v_{i-1} v_{j-1}$, $v_{j-1} v_j$ and $v_j v_{i-1}$. Let L be the line through v_j that is parallel to the line incident with v_{i-1} and v_{j-1} . It follows by the assumption that the vertices of P are not all on an open arc of the boundary of K lying between points of tangency of two parallel supporting lines that L intersects the circular arc $v_{j-1} v_i$ in a point v'_i . Let R'_1 denote the region bounded by the arcs $v_{i-1} v_{j-1}$, $v_{j-1} v'_i$ and $v'_i v_{i-1}$. It is clear that $\text{area}(R_1) \geq \text{area}(R'_1)$, and by the auxiliary statement proved above $\text{area}(R'_1) \geq \text{area}(R_2)$.

We obtain a sequence of double circle-polygons with non-decreasing areas and perimeters by iterating this construction until there are no sides with the property that the arc of the boundary of K assigned to one contains the arc assigned to the other. Thus, this process results in P^* in a finite number of steps. \square

3. CONCLUDING REMARKS

The theorems of Dowker and their extensions for the perimeter hold also on the sphere and in the hyperbolic plane. This was shown by Molnár [20] with the exception of the case concerning the perimeter of circumscribed polygons on the sphere. This last case was settled by L. Fejes Tóth [14] who observed that on the sphere the statements for the perimeter of circumscribed polygons and for the area of inscribed polygons are equivalent by spherical polarity.

With the exception of the case concerning the perimeter of circumscribed disc-polygons on the sphere the proofs of our theorems can be carried over to the sphere (for $r \leq \pi/2$) and the hyperbolic plane. The arguments deducing the theorems from Lemma 1 and Lemma 2 do not use the special structure of the geometry. The proofs of the case of area in Lemma 1 and the case of perimeter in Lemma 2 carry over to the hyperbolic plane and the sphere without change. We outline the changes needed in the proofs of the case of perimeter of Lemma 1 in the hyperbolic plane and the case of area of Lemma 2 in the hyperbolic plane and on the sphere.

Proof of the case of perimeter in Lemma 1 in the hyperbolic plane. In this argument we use the same notations as on Figure 1. On the hyperbolic plane we have $\angle po_i v_i = 2 \operatorname{arccot} \cosh r \tan(\frac{1}{2} \angle o_{i-1} po_i)$, $\angle po_i v''_j = 2 \operatorname{arccot} \cosh r \tan(\frac{1}{2} \angle o_i po')$, $\angle pov''_i = 2 \operatorname{arccot} \cosh r \tan(\frac{1}{2} \angle o_{i-1} po)$, and $\angle pov_j = 2 \operatorname{arccot} \cosh r \tan(\frac{1}{2} \angle o po')$. Hence

$$\begin{aligned} (1^*) \quad \angle v_i o_i v''_j &= \angle po_i v_i - \angle po_i v''_j \\ &= 2 \operatorname{arccot} \cosh r \tan(\angle o_{i-1} po_i / 2) - 2 \operatorname{arccot} \cosh r \tan(\angle o_i po' / 2) \end{aligned}$$

and

$$\begin{aligned} (2^*) \quad \angle v''_i o v_j &= \angle pov''_i - \angle pov_j \\ &= 2 \operatorname{arccot} \cosh r \tan(\angle o_{i-1} po / 2) - 2 \operatorname{arccot} \cosh r \tan(\angle o po' / 2). \end{aligned}$$

It is easily seen that the function $y = \operatorname{arccot}(\cosh r \tan x)$ is increasing and convex. We have

$$\angle o_{i-1} po_i / 2 - \angle o_i po' / 2 = -\angle o_{i-1} po' / 2 = \angle o_{i-1} po / 2 - \angle o po' / 2$$

and

$$\angle o_{i-1}po \geq \angle o_{i-1}po_i.$$

It follows that

$$\angle v_i o_i v_j'' \geq \angle v_i'' o v_j,$$

and similarly

$$\angle v_i o_i v_j'' \geq \angle v_i'' o v_j.$$

From here the proof of the inequality

$$\varrho_r(v_i', v_i) + \varrho_r(v_i, v_j') \geq \varrho_r(v_i', v_j) + \varrho_r(v_j, v_j')$$

follows in the same way as for the Euclidean plane.

Proof of the case of area in Lemma 2 on the sphere and on the hyperbolic plane.

The main tool in the Euclidean argument is the technical statement (*). We will show how direct analogues of (*) can be proved on the sphere and on the hyperbolic plane.

The locus of the points w' for which the triangles uvw and uvw' have the same orientation and area ($uvw' = \text{area}(uvw)$) is a constant distance curve for the line M through w , that is, a hypercycle on the hyperbolic plane and the so called Lexell circle on the sphere. Thus the role of the lines L_1 and L_2 are played by hypercycles and small circles, respectively. We introduce coordinates so that the origin is the intersection of M and the perpendicular bisector of the segment uv , the y coordinate of a point p is the signed distance from p to the orthogonal projection p_M of p on M , and the x coordinate of p is the signed distance from p_M to the origin. Let α denote the central angle of a circular segment of radius r cut off by a chord of length l . Then we have

$$s(l) = \begin{cases} \pi - \alpha \cos r - 2 \operatorname{arccot} \left(\frac{\cos r}{\cot \frac{\alpha}{2}} \right), & \alpha = 2 \arcsin \left(\frac{\sin \frac{l}{2}}{\sin r} \right) \\ \text{on the sphere,} \\ \alpha \cosh r + 2 \operatorname{arccot} \left(\frac{\cosh r}{\cot \frac{\alpha}{2}} \right) - \pi, & \alpha = 2 \arcsin \left(\frac{\sinh \frac{l}{2}}{\sinh r} \right) \\ \text{on the hyperbolic plane,} \end{cases}$$

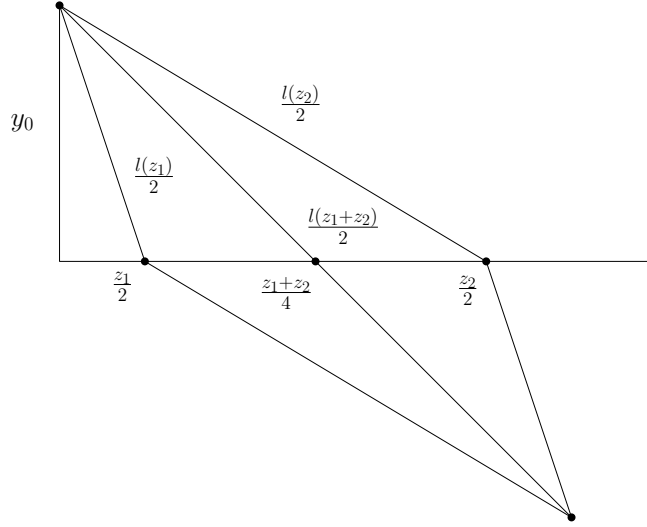
and

$$l(z) = \begin{cases} 2 \arccos \left(\cos \frac{z}{2} \cos y_0 \right) & \text{on the sphere,} \\ 2 \operatorname{arcosh} \left(\cosh \frac{z}{2} \cosh y_0 \right) & \text{on the hyperbolic plane.} \end{cases}$$

It can be checked by direct calculation, which is straightforward but tedious, that both $s(l)$ and $l(z)$ are strictly monotonically increasing and convex in both geometries. The anonymous referee suggested the following alternate argument to verify the monotonicity and convexity of $s(l)$ and $l(z)$.

The function $l(z)$ is twice the length of the hypotenuse of a right triangle whose catheti have lengths y_0 and $z/2$, and where y_0 is fixed. If we draw two such right triangles in a way that they share the right angle and the leg of length y_0 , cf. Figure 4, then the monotonicity of l is clear from the fact that in a triangle the larger side is opposite to the larger angle. Convexity of l is a consequence of the theorem that the length of a median of a triangle is less than the mean of the sides sharing a vertex with it. (This follows easily from the triangle inequality applying a central reflection in the midpoint of the third side of the triangle.)

The convexity of $s(l)$ can be proved as follows. If $K = \pm 1$ is the sectional curvature of the plane, then the sides a, b, c of a right triangle, where $c (\leq \pi/2$ in

FIGURE 4. The function $l(z)$.

the spherical case) is the hypotenuse, are related to one another by the formula

$$\cos(\sqrt{K}a) \cos(\sqrt{K}b) = \cos(\sqrt{K}c).$$

If we fix the hypotenuse c and express b as a function $b = f_c(a)$ of $a \in [0, c]$, then by differentiating the equation

$$\cos(\sqrt{K}a) \cos(\sqrt{K}f_c(a)) = \cos(\sqrt{K}c)$$

with respect to a , we obtain that

$$(3) \quad f'_c(a) = -\frac{\tan(\sqrt{K}a)}{\tan(\sqrt{K}f_c(a))}.$$

This yields that $f'_c(a)$ is negative for all $a \in [0, c]$, and hence, f_c is strictly decreasing on $[0, c]$. By differentiating (3) one more time with respect to a and using the monotonicity of the tangent and hyperbolic tangent functions, we obtain that f'_c is also strictly monotonically decreasing on $[0, c]$, which yields that f_c is concave.

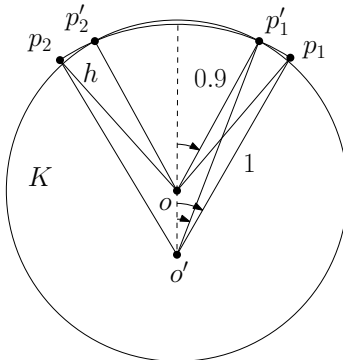
Denote by $p(r) = 2\pi \frac{\sin(\sqrt{K}r)}{\sqrt{K}}$ the perimeter of a circle of radius r and by $a_r(l)$ the length of the shorter arc of this circle cut off by a chord of length l . Consider a circle of perimeter $p(r)$ in the Euclidean plane and denote by $W_r(l)$ the length of the chord of this circle that cuts off an arc of length $a_r(l)$.

The area $s(l)$ is half the area of the intersection of two discs of radius r that share a chord of length l . Thus, using Theorem 5.1 of Csikós [7], we obtain that

$$s'(l) = -f'_r(l/2)W_r(l).$$

Since W_r is positive and strictly increasing and f'_r is negative and decreasing, it follows that s' is positive and increasing, which, in turn, yields that s is increasing and convex.

Now we can finish the proof in the same way as in the Euclidean case with the only difference that the role of the line L is played by the constant distance curve for the line $v_{i-1}v_{j-1}$ through v_j .



For two convex discs K_1 and K_2 let their *area deviation* $\delta_a(K_1, K_2)$ and *perimeter deviation* $\delta_p(K_1, K_2)$ be defined as

$$\delta_a(K_1, K_2) = \text{area}(K_1 \cup K_2) - \text{area}(K_1 \cap K_2)$$

$$\delta_p(K_1, K_2) = \text{per}(K_1 \cup K_2) - \text{per}(K_1 \cap K_2).$$

For a convex disc K let $d_a(n)$ and $d_p(n)$ denote the minimum area deviation and the minimum perimeter deviation of an n -gon from K , respectively. Eggleston [10] proved that both $d_a(n)$ and $d_p(n)$ are convex functions of n . It is an interesting question whether the minimum area deviation and the minimum perimeter deviation of a disc- n -gon of radius r from a hyperconvex disc K are also convex functions of n . To answer this question seems to be difficult. As one source of the difficulty in the case of the perimeter deviation we mention the following. Eggleston [10] proved that for a convex disc K among all convex n -gons the one closest to K in the sense of perimeter deviation is always inscribed in K . The following example shows that if K is a hyperconvex disc, then its best approximating disc- n -gon in the sense of perimeter deviation may neither be inscribed in nor circumscribed about K . Let K be the closed circular disc of radius 0.9 centred at the origin o . Let p_1 be an arbitrary point with $d(p_1, o) = h$ for some arbitrary fixed $0 < h < 1$. Then there exists a unique regular disc-pentagon $P_5(h)$ of radius 1 centred at o with one of its vertices equal to p_1 . Note that the disc-pentagon $P_5(0.9)$ is inscribed in K . Let h_1 be the value of h for which $P_5(h_1)$ is circumscribed about K .

Theorem 2 implies, on one hand, that $p_i(5) = \text{per}(P_5(0.9)) = 5.565\dots$, on the other hand, that $p_c(5) = \text{per}(P_5(h_1)) = 5.690\dots$. Hence $\delta_p(P_5(0.9), K) = 1.8\pi - 5.565\dots = 0.080\dots$, $\delta_p(P_5(h_1), K) = 5.690\dots - 1.8\pi = 0.04\dots$. Thus, $P_5(h_1)$ is closer to K than $P_5(0.9)$ in the sense of perimeter deviation. Now, we will examine the case when $h \in (0.9, h_1)$. Let the vertices of $P_5(h)$ be labeled in the positive direction and let p'_1 and p'_2 be the intersection points of the unit radius arc p_1p_2 and $\text{bd } K$ as shown on Figure 5. Let $\alpha = \angle oo'p_1$, $\beta = \angle oo'p'_1$, and $\gamma = \pi - \angle p'_1oo'$, cf. Figure 5. It is clear that

$$\begin{aligned} \delta_p(P_5(h), K) &= \text{per}(P_5(h) \cup K) - \text{per}(P_5(h) \cap K) \\ (4) \qquad \qquad &= (10(\alpha - \beta) + 9 \cdot \gamma) - (10\beta + (2\pi - 10\gamma) \cdot 0.9). \end{aligned}$$

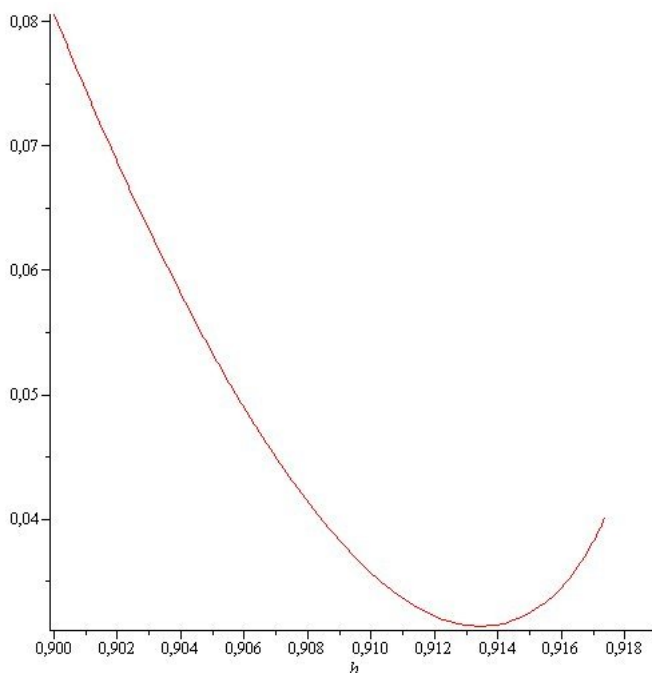


FIGURE 6. The graph was drawn by Maple 13.0

The right hand side of (4) can be expressed explicitly in terms of h using basic trigonometry; we leave the detailed calculations to the reader. The graph of (4) is shown on Figure 6.

It is apparent from Figure 6 that there is a whole subinterval of positive length of $(0.9, h_1)$ in which $P_5(h)$ approximates K better than both $P_5(0.9)$ and $P_5(h_1)$ in the sense of perimeter deviation. Thus the best approximating disc-pentagon of unit radius of K is neither inscribed in nor circumscribed about K .

4. ACKNOWLEDGEMENTS

The authors thank the anonymous referee whose suggestions have greatly improved the article. In particular, the arguments that show the monotonicity and convexity of the functions $s(l)$ and $l(z)$ were suggested by the referee.

The first author was supported by Hungarian OTKA grant K76154 and by the Fields Institute for Research in Mathematical Sciences.

The second author was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences, Hungarian OTKA grant K75016, and by the Fields Institute for Research in Mathematical Sciences.

REFERENCES

- [1] K. Bezdek, *Lectures on sphere arrangements—the discrete geometric side*, Fields Institute Monographs, vol. 32, Springer, New York, 2013.
- [2] K. Bezdek, *Classical topics in discrete geometry*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, New York, 2010.

- [3] K. Bezdek, R. Connelly, and B. Csikós, *On the perimeter of the intersection of congruent disks*, Beitrge Algebra Geom. **47** (2006), no. 1, 53–62.
- [4] K. Bezdek, Z. Lngi, M. Naszodi, and P. Papez, *Ball-polyhedra*, Discrete Comput. Geom. **38** (2007), 201–230.
- [5] E. Blanc, *Les ensembles surconvexes plans*, Ann. Sci. cole Norm. Sup. (3) **60** (1943), 215–246.
- [6] J. Buter, *berconvexe Mengen in der Ebene*, Proc. Akad. Wet. Amsterdam **41** (1938), 756–762.
- [7] B. Csikos, *On the volume of flowers in space forms*, Geom. Dedicata **86** (2001), 59–79.
- [8] L. Danzer, B. Grunbaum, and V. Klee, *Helly’s theorem and its relatives*, Proc. Sympos. Pure Math., Vol. VII, Amer. Math. Soc., Providence, R.I., 1963, pp. 101–180.
- [9] C. H. Dowker, *On minimum circumscribed polygons*, Bull. Amer. Math. Soc. **50** (1944), 120–122.
- [10] H. G. Eggleston, *Approximation to plane convex curves. (I) Dowker-type theorems*, Proc. London Math. Soc. (3) **7** (1957), 351–377.
- [11] G. Fejes Toth and L. Fejes Toth, *Remark on a paper of C. H. Dowker*, Period. Math. Hungar. **3** (1973), 271–274.
- [12] L. Fejes Toth, *Some packing and covering theorems*, Acta Sci. Math. Szeged **12** (1950), no. Leopoldo Fejer et Frederico Riesz LXX annos natis dedicatus, Pars A, 62–67.
- [13] L. Fejes Toth, *Remarks on polygon theorems of Dowker*, Mat. Lapok **6** (1955), 176–179 (Hungarian).
- [14] L. Fejes Toth, *Annherung von Eibereichen durch Polygone*, Math.-Phys. Semesterber. **6** (1958/1959), 253–261.
- [15] L. Fejes Toth, *Packing of r -convex discs*, Studia Sci. Math. Hungar. **17** (1982), 449–452.
- [16] L. Fejes Toth, *Packing and covering with r -convex discs*, Studia Sci. Math. Hungar. **18** (1982), 69–73.
- [17] Y. S. Kupitz, H. Martini, and M. A. Perles, *Finite sets in \mathbb{R}^d with many diameters — a survey*, Proceedings of the International Conference on Mathematics and Applications (ICMA-MU 2005, Bangkok), Mahidol University Press, Bangkok, 2005, pp. 91–112.
- [18] Y. S. Kupitz, H. Martini, and M. A. Perles, *Ball polytopes and the Vazsonyi problem*, Acta Math. Hungar. **126** (2010), 99–163.
- [19] A. E. Mayer, *Eine berkonvexitat*, Math. Z. **39** (1935), 511–531.
- [20] J. Molnar, *On inscribed and circumscribed polygons of convex regions*, Mat. Lapok **6** (1955), 210–218 (Hungarian).
- [21] L. Pasqualini, *Superconvexite*, Bull. Cl. Sci., Ser. V., Acad. R. Belg. **25** (1939), 18–24.
- [22] L. A. Santalo, *On plane hyperconvex figures*, Summa Brasil. Math. **1** (1946), 221–239 (Spanish).
- [23] J. G. van der Corput, *berkonvexe Mengen in der Ebene*, Proc. Akad. Wet. Amsterdam **41** (1938), 946–955.
- [24] P. Vincensini, *Sur les figures superconvexes planes*, Bull. Soc. Math. France **64** (1936), 197–208.

MTA ALFRED RENYI INSTITUTE OF MATHEMATICS, REALTANODA U. 13-15., 1053, BUDAPEST, HUNGARY

Email address: fejes.toth.gabor@renyi.mta.hu

DEPARTMENT OF GEOMETRY, BOLYAI INSTITUTE, UNIVERSITY OF SZEGED, ARADI VERTANUK TERE 1, 6720 SZEGED, HUNGARY, AND DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF CALGARY, CANADA

Email address: fodorf@math.u-szeged.hu